

On the higher-loop effective action in NJL model

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The 3-loop effective action and effective potential in Nambu - Jona-Lasinio model are calculated. The problem of vanishing contributions in the higher orders is discussed. The general form of such contributions is obtained.

1 Introduction

The generating functional of 1PI Green's functions, which hereinafter will be referred to as an effective action, is a convenient tool to derive various dynamical information from the system we want to investigate. Therefore it is important to have a method which makes it possible to calculate the higher-loop effective action for models with both bosonic and fermionic fields. Such a method based on the DeWitt formula [1] was developed in recent works [2] - [4], specifically, in reference to bosonized Nambu-Jona-Lasinio (NJL) model. The calculations were fulfilled up to 2-loop level. We want to continue using this method and, having made some generalizations, apply it to calculations in higher orders.

The effective action obtained will be used to find the effective potential for this model. Having calculated the effective potential, we can investigate the spontaneous breaking of chiral symmetry and the dynamical acquirement of mass by fermionic field [5] - [6]. The calculations carried out in [3] displayed an interesting property of NJL model, namely it turned out that 2-loop contribution to the effective potential vanishes. We shall discuss this problem in reference to higher orders and formulate and prove a theorem about vanishing contributions. Also we shall make the direct

calculations of 3-loop corrections to the effective potential and show that there exist non-vanishing corrections among them.

The article is organised as follows. In Section 2 we describe the method of calculations of the effective action. Section 3 is devoted to the calculation of 3-loop effective potential. In Section 4 we obtain the general form of vanishing contributions to the effective potential.

2 Method of calculations

In this section we shall review briefly the method based on the DeWitt formula. In order to demonstrate how it works we shall take QED but the results which will be obtained for QED are applicable after some substitutions to NJL model. This fact will be widely used in the next sections. We shall follow here the work [3] and only at the end of this section we make some generalisations which are important for higher-loop calculations.

The generating functional for the spinor electrodynamics with Lagrangian

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}[i(\gamma^\mu\partial_\mu - ie\gamma^\mu A_\mu) + m]\psi - \frac{1}{2\alpha}(\partial_\mu A^\mu)^2 \quad (1)$$

has the form (the source is introduced only for the photon field)

$$Z[J_\mu] = \int DAD\psi D\bar{\psi} \exp[i \int L dx + i(JA)] \quad (2)$$

Having integrated over spinor variables we get

$$Z[J_\mu] = \int DA \exp[iS_{eff} + i(JA)] \quad (3)$$

where

$$S_{eff} = S_A + i\hbar T \quad (4)$$

$$S_A = \frac{1}{2}A_\mu D_{\mu\nu}^{-1}A_\nu, \quad T = \text{Tr} \ln K, \quad (5)$$

$$K^{-1} = i\gamma^\mu\partial_\mu + e\gamma^\mu A_\mu - m, \quad D_{\mu\nu} = \frac{1}{\partial^2}(g_{\mu\nu} - (1 - \alpha)\frac{\partial_\mu\partial_\nu}{\partial^2})$$

Let us define the effective action in the following way

$$\Gamma[\langle A_\mu \rangle] = W[J_\mu] - (J_\nu \langle A^\nu \rangle) \quad (6)$$

where

$$\langle A_\mu \rangle = Z^{-1} \int D A A_\mu \exp[i S_{eff} + i(JA)] \quad (7)$$

is so-called "classical field" and

$$W[J_\mu] = -i \ln Z \quad (8)$$

The effective action defined this way is the generating functional of the Green's functions which are 1PI with respect to photon propagator. From (3) and (6) we have

$$\frac{\delta \Gamma}{\delta \langle A_\mu \rangle} = \langle \frac{\delta S_{eff}}{\delta A_\mu} \rangle \quad (9)$$

For the models containing fermion and vector fields the general form of DeWitt's formula is

$$\langle Q[A, \bar{\psi}, \psi] \rangle =: \exp(G) : Q[\langle A \rangle, \langle \bar{\psi} \rangle, \langle \psi \rangle] \quad (10)$$

where

$$G = \frac{i}{\hbar} \sum_{n=2}^{\infty} \frac{(-i\hbar)^n}{n!} \sum \frac{C_{ijk}^n}{\delta^n} (-1)^j G^{\mu_1 \dots \mu_i \alpha_1 \dots \alpha_j \beta_1 \dots \beta_k} \times \quad (11)$$

$$\times \frac{1}{\delta \langle A_{\mu_1} \rangle \dots \delta \langle \psi_{\beta_k} \rangle \dots \delta \langle \bar{\psi}_{\alpha_j} \rangle \dots}$$

$$G^{\mu_1 \dots \mu_i \alpha_1 \dots \alpha_j \beta_1 \dots \beta_k} = \frac{\delta^n W}{\delta J_{\mu_1} \dots \delta \eta_{\beta_k} \dots \delta \bar{\eta}_{\alpha_j} \dots} \quad (12)$$

are the connected Green's functions with i photon and $j+k$ fermion legs; the colons mean that derivatives act only on Q ; the Plank constant \hbar is restored and

$$C_{ijk}^n = \frac{n!}{i!j!k!}, \quad i + j + k = n \quad (13)$$

Having applied (10) to the rhs of (9) we get

$$\frac{\delta \Gamma}{\delta \langle A_\mu \rangle} =: \exp(G) : \frac{\delta S_{eff}[\langle A_\mu \rangle]}{\delta \langle A_\mu \rangle} \quad (14)$$

Below we shall deal only with $\langle A \rangle$ therefore for convenience we shall omit brackets keeping in mind that from now on A implies $\langle A \rangle$. In our case we have for G

$$G = \frac{-i\hbar}{2} G^{\mu\nu} \frac{\delta^2}{\delta A^\mu \delta A^\nu} - \frac{\hbar^2}{6} G^{\mu\nu\lambda} \frac{\delta^3}{\delta A^\mu \delta A^\nu \delta A^\lambda} + \frac{i\hbar^3}{24} G^{\mu\nu\lambda\sigma} \frac{\delta^4}{\delta A^\mu \delta A^\nu \delta A^\lambda \delta A^\sigma} \quad (15)$$

where

$$G^{\alpha_1 \dots \alpha_n} = \frac{\delta^n W}{\delta J_{\alpha_1} \dots \delta J_{\alpha_n}} \quad (16)$$

It is convenient to use the following notation

$$T^{\alpha_1 \dots \alpha_s} = (-1)^s \frac{\delta^s T}{\delta A_{\alpha_1} \dots \delta A_{\alpha_s}} \quad (17)$$

Then

$$\frac{\delta S_{eff}}{\delta A_\mu} = D^{-1}{}^{\mu\nu} A_\nu - i\hbar T^\mu \quad (18)$$

and we have from (14) up to the order of \hbar^3

$$\begin{aligned} \frac{\delta \Gamma}{\delta A_\mu} &= D^{-1}{}^{\mu\nu} A_\nu - \hbar \left[iT^\mu + \frac{\hbar}{2} G^{\nu\lambda} T_{\mu\nu\lambda} \right] - \\ &\hbar^2 \left[\frac{i\hbar}{6} G^{\nu\lambda\sigma} T_{\mu\nu\lambda\sigma} - \frac{i\hbar}{8} G^{\nu_1\lambda_1} G^{\nu_2\lambda_2} T_{\mu\nu_1\lambda_1\nu_2\lambda_2} \right] \end{aligned} \quad (19)$$

We also need an equation to connect G and Γ . This equation is

$$G^{\mu\nu} \frac{\delta^2 \Gamma}{\delta A^\nu \delta A^\lambda} = -\delta_\lambda^\mu \quad (20)$$

Now we expand G and Γ in series over \hbar

$$\begin{aligned} \Gamma &= \Gamma_0 + \hbar \Gamma_1 + \hbar^2 \Gamma_2 + \dots \\ G &= G_0 + \hbar G_1 + \hbar^2 G_2 + \dots \end{aligned} \quad (21)$$

Here Γ_0 is the tree effective action, and $\Gamma_1, \Gamma_2, \dots$ are the quantum corrections - 1-loop, 2-loop, ... accordingly. Up to the 2-loop level the effective action was obtained in [3]:

$$\Gamma = \frac{1}{2} A_\mu D^{-1}{}^{\mu\nu} A_\nu + i\hbar T + \frac{1}{2} \hbar^2 \text{Tr}(-D^{\mu\nu} T_{\mu\nu}) \quad (22)$$

Now we want to make some comments. The condensed notations used up to now imply summarizing over discrete variables and integrating over continuous ones in all the expressions. Nevertheless the question of how to calculate traces over γ -matrices was out of discussion. Therefore the method described above should be expanded with the appropriate rule in order to satisfy the common rules of diagrammatic technique (see for example [7]). Namely, the calculations of traces must be carried out along every fermionic loop separately (in our case the loops are constructed not out of the simple propagators but out of the propagators in "external field"). If we have only one loop this rule is satisfied by calculation of traces over the whole expressions, but in the case of many loops it is not so.

3 3-loop effective action

Now we start investigating NJL model. We take this model in the bosonized form

$$L = \bar{\psi}(i\gamma^\mu\partial_\mu + g(\sigma + i\gamma_5\pi))\psi - \frac{1}{2}(\sigma^2 + \pi^2) \quad (23)$$

where σ and π are auxiliary scalar and pseudoscalar fields. After integrating over spinor variables in the path integral we have the following generating functional for this model

$$Z[J] = \int D\phi \exp(iS_{eff} + i(J\phi)) \quad (24)$$

where we used the notations

$$\phi^i = \{\sigma, \pi\}, \quad \gamma^i = \{1, i\gamma_5\}, \quad \phi^2 = \sigma^2 + \pi^2 \quad (25)$$

and

$$\begin{aligned} S_{eff} &= -\frac{1}{2}\phi^2 + i\hbar T, \\ T &= \text{Tr} \ln K, \quad K^{-1} = i\gamma^\mu\partial_\mu + g\hat{\phi}, \quad \hat{\phi} = \phi^i\gamma^i \end{aligned} \quad (26)$$

Having introduced the free propagator of the auxiliary field ϕ^i as

$$D_{ij}(x - y) = \delta(x - y)\delta_{ij} \quad (27)$$

we can re-write S_{eff} as

$$S_{eff} = -\frac{1}{2}\phi^i D_{ij}^{-1} \phi^j + i\hbar T \quad (28)$$

It is easy to see that the NJL model re-written in such a form becomes analogous to QED after the substitutions

$$i, j, \dots, \phi^i \rightarrow \mu, \nu, \dots, A^\mu \quad (29)$$

Therefore we get the effective action for NJL model (up to 2-loop level)

$$\tilde{\Gamma} = -\frac{1}{2}\phi^i D_{ij}^{-1} \phi^j + i\hbar T + \frac{1}{2}\hbar^2 \text{Tr}(-D^{ij} T_{ij}) \quad (30)$$

The corresponding diagrams are represented in Fig.1. The auxiliary boson field is depicted by the dashed line, the thick lines are the fermionic "propagators in external field".

Here we want to make some remarks about what we are calculating. In the case of QED we implied according to (2) and (6) that there exist only photon external fields and the fermion propagators occur only as internal lines. After we turned to study of NJL model the effective action obtained in such a manner is the generating functional of the Green's functions with no free quarks in external lines. Of course, these Green's functions will be irreducible with respect to the propagators of the auxiliary field but actually we are not interested in this fact because these propagators will reduce into the points according to (27).

Let us calculate 3-loop effective action and effective potential. In order to do it we return to Eq.(19) which was obtained for QED and use the substitutions (29). We have

$$\frac{\delta \tilde{\Gamma}_3}{\delta \phi^i} = -\frac{1}{2} G_1^{jk} T_{ijk} - \frac{i}{6} G_0^{jkl} T_{ijkl} + \frac{i}{8} G_0^{j_1 k_1} G_0^{j_2 k_2} T_{ij_1 k_1 j_2 k_2} \quad (31)$$

G_0 and G_1 can be found from the following equation

$$\left(G_0^{lj} + \hbar G_1^{lj} + \dots \right) \left(\frac{\delta^2 \tilde{\Gamma}_0}{\delta \phi^j \delta \phi^k} + \hbar \frac{\delta^2 \tilde{\Gamma}_1}{\delta \phi^j \delta \phi^k} + \dots \right) = -\delta_j^l \quad (32)$$

From (31) and (32) we get

$$\frac{\delta \tilde{\Gamma}_3}{\delta \phi^i} = -\frac{i}{2} G_0^{jp} T_{pr} G_0^{rk} T_{ijk} - +\frac{i}{8} G_0^{j_1 k_1} G_0^{j_2 k_2} T_{ij_1 k_1 j_2 k_2} \quad (33)$$

To obtain the last equation we took into consideration that

$$G_0^{abc} = G_0^{aj} G_0^{bk} G_0^{cl} \frac{\delta^3 \tilde{\Gamma}_0}{\delta \phi^j \delta \phi^k \delta \phi^l} = 0 \quad (34)$$

on the strength of (30).

Therefore 3-loop contribution to the effective action for NJL model is

$$\begin{aligned} \tilde{\Gamma}_3 &= \frac{i}{4} G_0^{ik} T_{ij} G_0^{jl} T_{kl} + \frac{i}{8} G_0^{ij} G_0^{kl} T_{ijkl} \\ G_0^{ij} &= D^{ij} \end{aligned} \quad (35)$$

The corresponding diagrams are shown in Fig.2.

Let us remind the comments made at the end of Section 2. It is easy to see that in the left diagram in Fig.2 traces must be calculated over each fermionic loop separately and this rule will not be changed when we take into consideration that propagator D^{ij} of the auxiliary field assembles into a point according to (27) (Fig.3). Therefore the contributions corresponding to left and right diagrams in Fig.3 are not equivalent.

The effective potential can be obtained from the the effective action by setting all the "classical fields" equal to constants. Thus we have

$$\tilde{\Gamma} = - \int dx V_{NJL} \quad (36)$$

Let us write down separately the contributions corresponding to each graphs in Fig.3 ($I_{(a)}$ represents left, and $I_{(b)}$ and $I_{(c)}$ accordingly central and right ones)

$$\begin{aligned} I_{(a)} &= g^4 \int \prod_{i=1}^4 \frac{d^d p_i}{(2\pi)^d} \delta(\sum p_i) \text{Tr} \left[(\gamma^\mu p_{1\mu} + g\hat{\phi})^{-1} \gamma^i (\gamma^\mu p_{2\mu} + g\hat{\phi})^{-1} \gamma^j \right] \times \\ &\times \text{Tr} \left[(\gamma^\mu p_{3\mu} + g\hat{\phi})^{-1} \gamma^i (\gamma^\mu p_{4\mu} + g\hat{\phi})^{-1} \gamma^j \right] \end{aligned} \quad (37)$$

$$I_{(b)} = g^4 \int \prod_{i=1}^4 \frac{d^d p_i}{(2\pi)^d} \delta(p_2 + p_3) \text{Tr} \left[(\gamma^\mu p_{1\mu} + g\hat{\phi})^{-1} \gamma^i (\gamma^\mu p_{2\mu} + g\hat{\phi})^{-1} \gamma^i \times \right. \\ \left. \times (\gamma^\mu p_{3\mu} + g\hat{\phi})^{-1} \gamma^j (\gamma^\mu p_{4\mu} + g\hat{\phi})^{-1} \gamma^j \right] \quad (38)$$

$$I_{(c)} = g^4 \int \prod_{i=1}^4 \frac{d^d p_i}{(2\pi)^d} \delta(\sum p_i) \text{Tr} \left[(\gamma^\mu p_{1\mu} + g\hat{\phi})^{-1} \gamma^i (\gamma^\mu p_{2\mu} + g\hat{\phi})^{-1} \gamma^j \times \right. \\ \left. \times (\gamma^\mu p_{3\mu} + g\hat{\phi})^{-1} \gamma^i (\gamma^\mu p_{4\mu} + g\hat{\phi})^{-1} \gamma^j \right] \quad (39)$$

(d is the dimension of space-time; we omit $\int dx$)

In order to calculate the traces we use the following representation

$$(\gamma^\mu p_\mu + g\hat{\phi})^{-1} = (\gamma^\mu p_\mu - \frac{g}{p^2} \gamma^\mu p_\mu \hat{\phi} \gamma^\nu p_\nu) (p^2 - g^2 \phi^2)^{-1} \quad (40)$$

which can be easily checked if we use

$$\hat{\phi}(\gamma^\mu p_\mu) \hat{\phi}(\gamma^\nu p_\nu) = \phi^2 p^2 \quad (41)$$

After summarizing the traces there will be only the contributions from $I_{(a)}$ and $I_{(c)}$ so we have the final form of 3-loop effective potential

$$V_3 = -\frac{1}{2} d^2 g^4 \int \left(\prod_{i=1}^4 \frac{d^d p_i}{(2\pi)^d} (p_i^2 - g^2 \phi^2)^{-1} \right) (2\pi)^d \delta(\sum p_i) (p_1 \cdot p_2) (p_3 \cdot p_4) \\ - \frac{1}{2} d^2 g^8 \phi^4 \int \left(\prod_{i=1}^4 \frac{d^d p_i}{(2\pi)^d} (p_i^2 - g^2 \phi^2)^{-1} \right) (2\pi)^d \delta(\sum p_i) \\ - \frac{3}{2} d g^6 \phi^2 \int \left(\prod_{i=1}^4 \frac{d^d p_i}{(2\pi)^d} (p_i^2 - g^2 \phi^2)^{-1} \right) (2\pi)^d \delta(\sum p_i) [(p_1 \cdot p_2) + (p_3 \cdot p_4)] \quad (42)$$

In order to make these integrals finite we should introduce a cut-off. In the case of $d = 2$ as the theory is renormalizable the cut-off has no physical sense but if $d = 4$ the cut-off becomes a new phenomenological parameter of the theory. It may be fixed by the various normalization conditions (see for example [8]).

The fact that $I_{(b)}$ is vanishing is the consequence of some general properties of the model we have chosen. As a matter of fact, there exist a species of vanishing contributions in the effective potential. In the next section we shall formulate and prove the theorem which gives these contributions in the most general form.

4 Vanishing contributions

Theorem. *The contributions to the effective potential of NJL model given by the expressions like*

$$I = \int d^d p_1 d^d p_2 \{d^d q\} \delta(f(q)) \times \\ \times \text{Tr} \left[\gamma^i (\gamma^\mu p_{1\mu} + g\hat{\phi})^{-1} \gamma^i \dots F(\{q\}) \dots (\gamma^\mu p_{2\mu} + g\hat{\phi})^{-1} \right] \quad (43)$$

are equal to zero.

The graphic representations some of such contributions are shown in Fig.4. As is seen, the necessary conditions for diagrams to be vanishing are:

- 1) presence of at least 2 loops with independent momenta within them (loops are constructed out of the "propagators in external field");
- 2) traces is calculated over the whole expression, not over any part separately.

In order to prove it we use first the fact that momenta p_1 and p_2 are not mixed with the other momenta $\{q\}$ in the δ -function. Let us re-write (43) as

$$I = \text{Tr} \left[\gamma^i \left(\int d^d p_1 (\gamma^\mu p_{1\mu} + g\hat{\phi})^{-1} \right) \gamma^i \times \right. \\ \left. \times \int \{d^d q\} F(\{q\}) \left(\int d^d p_2 (\gamma^\mu p_{2\mu} + g\hat{\phi})^{-1} \right) \right] \quad (44)$$

On the strength of (40) we get (d is 2 or 4)

$$\int d^d p (\gamma^\mu p_\mu + g\hat{\phi})^{-1} = \int d^d p (\gamma^\mu p_\mu - \frac{g}{p^2} \gamma^\mu p_\mu \hat{\phi} \gamma^\nu p_\nu) (p^2 - g^2 \phi^2)^{-1} = \\ = \int d^d p (-\frac{g}{p^2}) \gamma^\mu p_\mu \hat{\phi} \gamma^\nu p_\nu (p^2 - g^2 \phi^2)^{-1} \quad (45)$$

Further, as the cyclic permutations do not change the trace, we have

$$I = \text{Tr} \left[\int \{d^d q\} F(\{q\}) \int d^d p_1 \int d^d p_2 \right. \\ \left. (\gamma^\mu p_{2\mu} \hat{\phi} \gamma^\nu p_{2\nu}) \gamma^i (\gamma^\mu p_{1\mu} \hat{\phi} \gamma^\nu p_{1\nu}) \gamma^i \frac{g^2}{p_1^2 p_2^2} (p_2^2 - g^2 \phi^2)^{-1} (p_1^2 - g^2 \phi^2)^{-1} \right] \quad (46)$$

Keeping in mind that $\gamma^i = \{1, i\gamma_5\}$ we get

$$(\gamma^\mu p_{2\mu} \hat{\phi} \gamma^\nu p_{2\nu}) \gamma^i (\gamma^\mu p_{1\mu} \hat{\phi} \gamma^\nu p_{1\nu}) \gamma^i = (\sigma - i\gamma_5 \pi) \gamma^i (\sigma - i\gamma_5 \pi) \gamma^i p_1^2 p_2^2 = \\ = (\sigma - i\gamma_5 \pi) [(\sigma - i\gamma_5 \pi) + \sigma (i\gamma_5)^2 - i\gamma_5 (i\gamma_5)^2 \pi] p_1^2 p_2^2 = 0 \quad (47)$$

Thereby we have proved that $I = 0$. The proof is completed.

There is an obvious way to generalize the theorem. Namely, *the expressions like*

$$\tilde{I} = \int d^d p_1 d^d p_2 \{d^d q\} \text{Tr} [\dots] \dots \text{Tr} [\dots] \quad (48)$$

are equal to zero if at least one of the traces in the integrand can be reduced to the form given by the Eq.(42). An example is shown in Fig.5.

The only graph which gives contribution to 2-loop effective potential is the 2-loop diagram in Fig.4 (this graph can be derived from the right graph in Fig.1 after reducing the propagator of the auxiliary field). Obviously, this is the very case we discussed in the theorem above because the analytical expression for this graph has the form

$$\int \frac{d^d p_1}{(2\pi)^d} \frac{d^d p_2}{(2\pi)^d} (\gamma^\mu p_{1\mu} + g\hat{\phi})^{-1} (\gamma^\mu p_{2\mu} + g\hat{\phi})^{-1} \quad (49)$$

Therefore there exist no contributions of the order \hbar^2 in the effective potential. Another example is our Eq.(38) which determines contribution $I_{(b)}$ to 3-loop effective potential. On the other hand, there exist other contributions in higher (> 2) orders than those represented in Fig.4, and as was shown by the direct calculations for the order of \hbar^3 there are non-vanishing ones among them.

5 Conclusion

In this work we used the method based on the DeWitt formula to calculate the effective action and the effective potential for bosonized NJL model. Recently it was shown that 2-loop effective potential in this model is equal to zero. As the value of effective potential is the subject of importance when study phenomena like spontaneous breaking of chiral symmetry it would be very attractive if the higher contributions behaved themselves the same way, i.e. were vanishing too, because in this case the (tree + 1-loop) effective potential would be the precise result. But 3-loop calculations revealed non-vanishing contributions which of course must be taken into consideration as well as (if necessary) the higher (4-, 5-,...loop) ones.

The other interesting problem is which contributions are vanishing besides 2-loop one. The most general form of these contributions was obtained (Eqs (43) and (48)) and discussed in the special theorem in Section 4. As was shown, the crucial conditions for contributions to be vanishing are the independence of the momenta within at least 2 loops and the manner of calculation of traces over γ -matrices.

References

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Figures 1-3

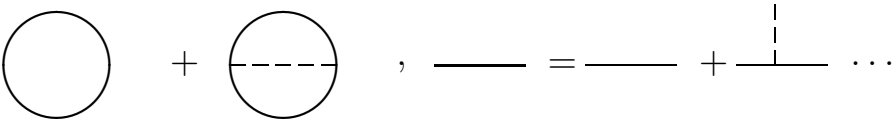


Fig.1

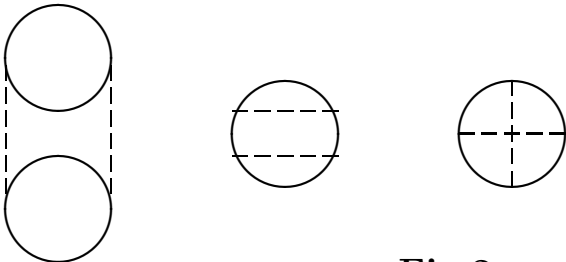


Fig.2

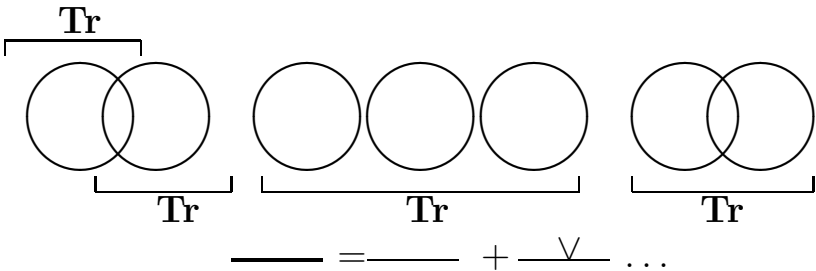


Fig.3

Figures 4-5

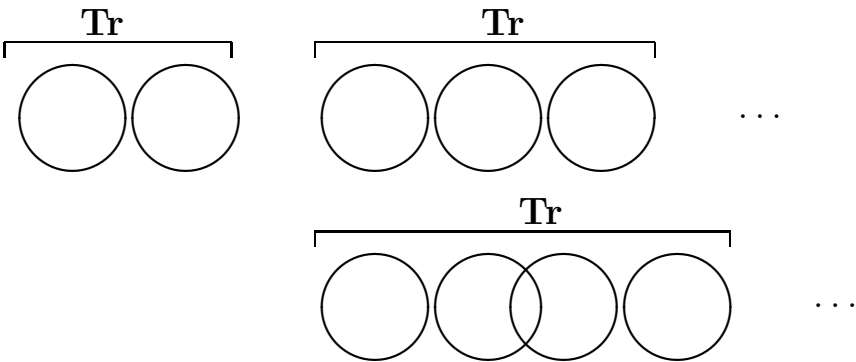


Fig.4

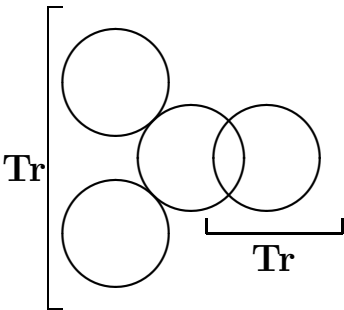


Fig.5